Understanding subsonic and supersonic nozzle flow using the NAG Library

Subsonic and supersonic flow of perfect gas through orifices is a topic which is well studied by a number of researchers. Orial Kryeziu, University College London is currently considering the steady, isentropic, two-dimensional compressible flow from a nozzle for fluid velocities in both the subsonic and supersonic ranges. However the larger goal of his research is to have a clear understanding of incompressible fluid flow through gaps or slits modeled using the shallow water equations. Working with compressible instead of incompressible fluid is possible because of well established links between the shallow water equations and gas dynamic equations. Gas equations and methods developed to solve them should be useful for studying the geophysical dynamics of incompressible fluid through gaps.

A widely used technique for analysing steady two-dimensional, irrotational, compressible, homentropic flow is the hodograph method. The hodograph method is based on the idea of regarding position `{x}=(x,y)` as a function of the velocity `{u}=(u,v)` and determining the partial differential equations (PDE) satisfied by `{x}`. This transformation between variables corresponds to a general type of transformation called the Legendre transfomation. An advantage of the hodograph transformation is that hodograph equations are linear and have solutions readily available by analytical and numerical techniques.

P. Cook and E. Newman formulated the nozzle problem in the hodograph plane, where the continuity equation was expressed in terms of the Legendre potential. In two dimensions this PDE is linear and enjoys the special property that the equation is elliptic in the subsonic region of the flow field and hyperbolic in the supersonic region of the flow field. Transformation to the hodograph coordinates maps the subsonic region into a rectangle, thus allowing the use of a finite difference method with a rectangular mesh.

In 1902, Chaplygin gave an analytical solution of subcritical compressible flow from an infinite reservoir using the hodograph transformation. The system of equations expressing irrotationality and conservation of mass was transformed into a PDE (Chaplygin's equation) for the stream function. Solutions of Chaplygin's equation, and hence of the original fluid dynamic equations, do exist in the form of infinite series of hypergeometric functions provided that the critical pressure ratio is not exceeded. This solution cannot be extended to supercritical flow.

As pressure ratio is reduced, jet speed from the nozzle becomes supersonic. Chaplygin's equation is hyperbolic for these speeds. The method of characteristics is the most accurate numerical technique for solving hyperbolic PDEs. An important restriction to keep in mind when working with the method of characteristics is that the dependent variables must be continuous in the region of interest. This requirement limits the applicability of the method of characteristics to regions where all of the dependent variables are continuous, but excludes regions where there are discontinuities, such as shock waves.

Due to difficulties associated with obtaining continuous solutions in a flow field containing a subsonic and a supersonic region, separate calculations in each region are performed. The common boundary condition along the sonic line is adjusted to give a continuous solution; Norwood described such a technique for obtaining solutions to supercritical flow. Benson and Pool described similar numerical solutions and gave quantitative detail of the structure of the jet by carrying out analytical and experimental investigation.

In order to solve the governing equations Orial turned to the specialised numerical algorithms provided by NAG. Chaplygin's equation, in terms of the streamfunction, was solved using the NAG routine D03EDF which implements a multigrid technique. The boundary value problem was transformed in a seven-diagonal system of linear equations arising from the discretization of the PDE and the boundary conditions. Special suited utility subroutines, part of the NAG routine, allowed me to construct the problem quickly and with minimum effort.

Solution to the problem in terms of the Legendre potential was not possible to achieve using the same routine, because the boundary conditions caused the iterate to diverge. This obstacle was easily resolved by using another NAG routine that implemented a direct rather than iterative method.

Orial said of his use of NAG *“Having not used numerical algorithms such as those developed by NAG before, I found it easy to understand its usage and how to integrate routines into by own application program. I believe NAG will continue to be a vital tool of my work. It will allow me to concentrate on my work without worrying about long established numerical methods.”*