# *Stochastic Grid Bundling Method for Bermudan Swaptions*

## *In Simple LIBOR Test Case, SGBM is 4x–6x Faster than Longstaff–Schwartz for Similar Accuracy*

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## Bermudan Options

Fix times $0 = T_0 < T_1 < \ldots < T_M = T$ and let $(S_t)_{t\geq 0}$ denote a risky asset. The value of a Bermudan option with payoff $h$ is defined as

$$V(T_0, S(T_0)) = \max_{\tau \in \mathcal{T}} \mathbb{E}\left[\frac{h(S(\tau))}{B(\tau)}\right]$$

where $(B_t)_{t\geq 0}$ is a bank account and $\mathcal{T}$ is all $\{T_m\}_{m=0}^{M}$ valued stopping times. Determining $V$ requires finding the *continuation value*

$$Q(T_m, S(T_m)) := \mathbb{E}\left[\frac{B(T_m)}{B(T_{m+1})} V(T_{m+1}, S(T_{m+1})) \,\middle|\, S(T_m)\right]$$

## Longstaff–Schwartz Method (LSM)

This method determines $Q$ by choosing basis functions $\phi_1, \ldots, \phi_K$ and solving the regression equation

$$\frac{B(T_m)}{B(T_{m+1})} V(T_{m+1}, S(T_{m+1})) \approx \sum_{k=1}^{K} \beta_k \phi_k(T_m, S(T_m))$$

for coefficients $\beta_1, \ldots, \beta_K$ at each time step $T_m$. This amounts to making two approximations:

- $V$ is approximated by a linear combination of basis functions **on the entire path space**
- $S(T_{m+1})$ is approximated by $S(T_m)$

## Stochastic Grid Bundling Method (SGBM)

At each time step $T_m$, SGBM partitions the path space into a number of "bundles". Within each bundle it makes the regression approximation

$$V(T_{m+1}, S(T_{m+1})) \approx \sum_{k=1}^{K} \beta_k \phi(T_{m+1}, S(T_{m+1})).$$

- Each regression defines a **local approximation** on its bundle, thus avoiding making a global approximation over the whole path space. This means fewer basis functions are needed.
- The regression only uses data at time $T_{m+1}$, thus avoiding the second Longstaff–Schwartz approximation $S(T_{m+1}) \approx S(T_m)$

Using the equation above gives

$$Q(T_m, S(T_m)) = \mathbb{E}\left[\frac{B(T_m)}{B(T_{m+1})} V(T_{m+1}, S(T_{m+1}) \,\middle|\, S(T_m)\right]$$

$$\approx \mathbb{E}\left[\frac{B(T_m)}{B(T_{m+1})} \sum_{k=1}^{K} \beta_k \phi_k(T_{m+1}, S(T_{m+1})) \,\middle|\, S(T_m)\right]$$

$$= \sum_{k=1}^{K} \beta_k \mathbb{E}\left[\frac{B(T_m)}{B(T_{m+1})} \phi_k(T_{m+1}, S(T_{m+1})) \,\middle|\, S(T_m)\right]$$

$$:= \sum_{k=1}^{K} \beta_k \psi_k(T_m, S(T_m))$$

where $\psi_k$'s are conditional expectations of discounted basis functions.

## The Test Case: Bermudan Swaption on LIBOR

We considered a Bermudan Swaption (10yr maturity, 4yr reset, semi-annual payments and exercises) driven by a one factor LIBOR Market Model (LMM).

### Basis Functions

For Longstaff–Schwartz we took three basis functions $\phi_1 = 1$, $\phi_2 =$ the next-to-maturity swap rate, and $\phi_3 = \phi_2^2$. This gave sufficient accuracy. For SGBM it proved sufficient to take only $\phi_1$ and $\phi_2$.

### Computing $\psi_2$

SGBM requires $\psi_2$, the conditional expectation of the discounted next-to-maturity swap rate. Under LMM there is no analytic formula for this. We approximated $\psi_2$ with the swap rate approximation formula introduced by Rebonato. Accuracy was tested with nested Monte Carlo and found to be fairly accurate, although on average low-biased.

### Bundling

Partitioning the multidimensional LMM path space would be prohibitively expensive. Instead we used $\psi_2$ to map the path space onto $\mathbb{R}$ and bundled this one-dimensional space.

## Results of the Test Case

We considered two price estimators: the **direct estimator** produced by the regression framework, and a **path estimator** applying this exercise strategy to a new set of paths. The path estimator is a lower bound.

Figure 1 shows that **SGBM consistently produces better exercise strategies** (path estimator) than Longstaff–Schwartz.

Figure 2 shows that the **SGBM direct estimator has less than half the standard error** of Longstaff–Schwartz.

Figure 3 plots overall runtimes (LIBOR path generation + direct estimator CPU time) vs standard error. Although the SGBM direct estimator is roughly 3x more expensive than Longstaff–Schwartz, it needs roughly 10x fewer paths to achieve a given standard error. Since LIBOR sample paths are fairly expensive to generate, the result is that the **SGBM direct estimator is between 4x and 6x faster overall for a given standard error**.

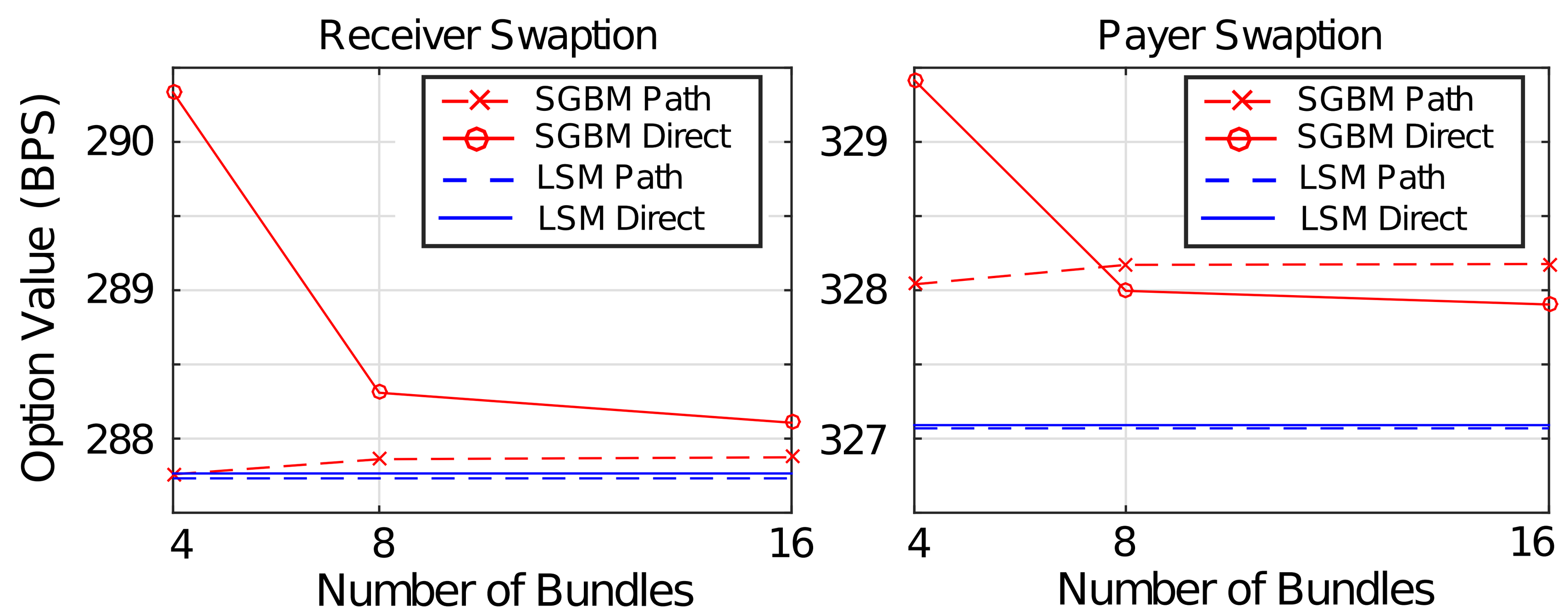


*Figure 1*: Price vs number of bundles. Note the SGBM path estimator is consistently better (higher) than Longstaff–Schwartz.

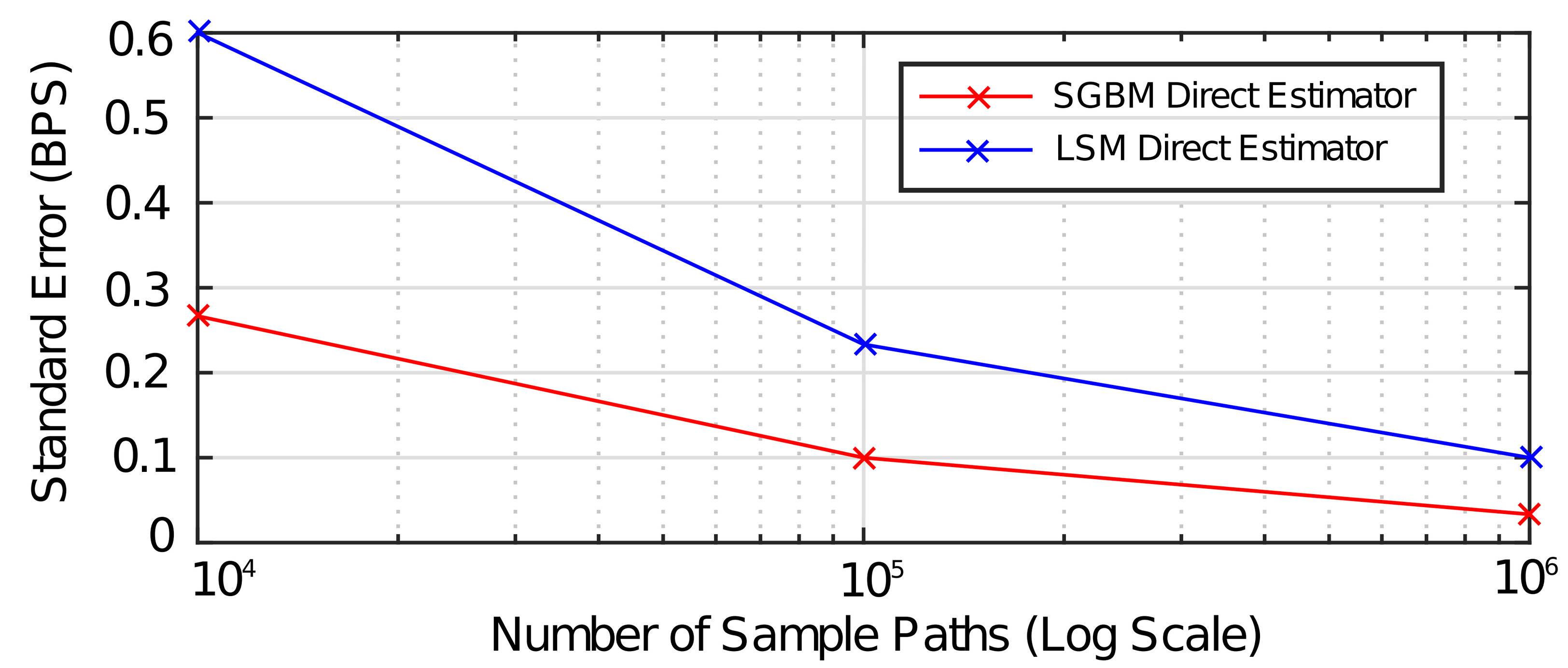


*Figure 2*: Standard errors of SGBM (8 bundles) vs Longstaff–Schwartz. Note Longstaff–Schwartz needs roughly 10x more paths to achieve the same standard error.

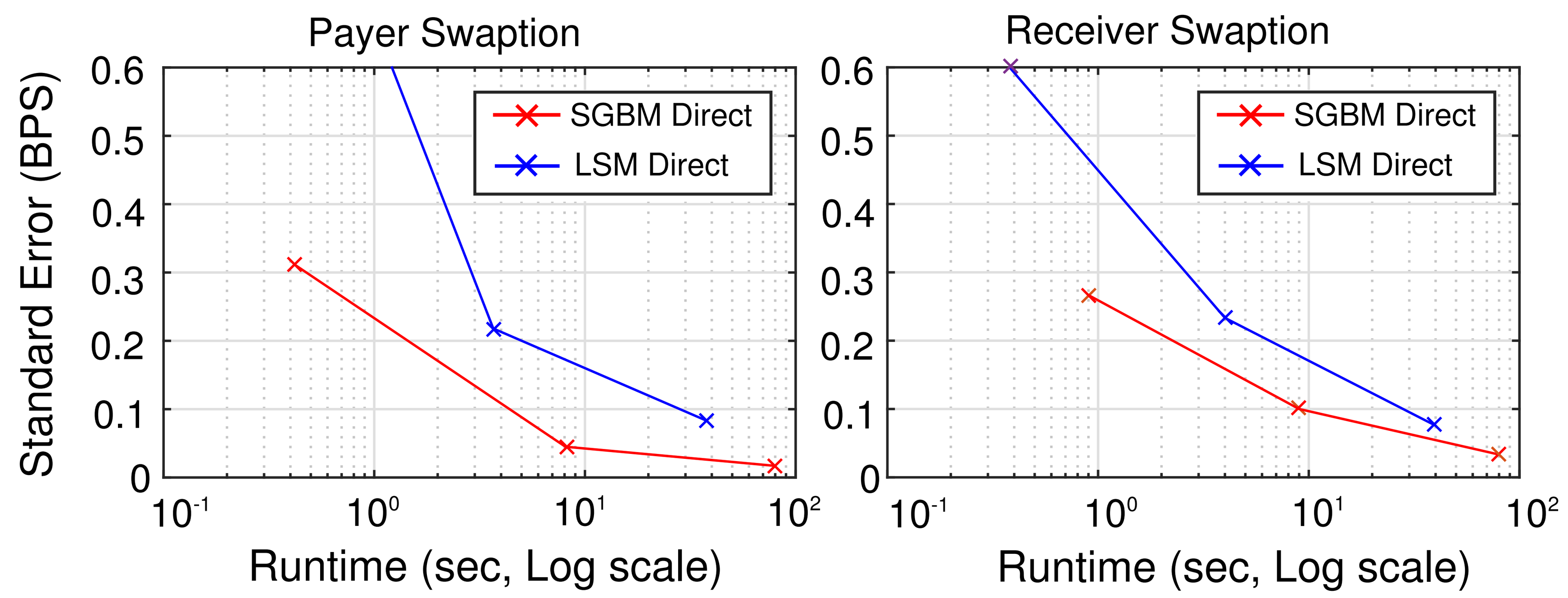


*Figure 3*: Overall runtime (LIBOR path generation + direct estimator) for SGBM (8 bundles) vs Longstaff–Schwartz for sample sizes $n = 10^4, 10^5, 10^6$. For a given standard error, the SGBM direct estimator is between 4x and 6x faster than Longstaff–Schwartz.

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