NAG Library Function Document

nag_bessel_y1 (s17adc)

1 Purpose

nag_bessel_y1 (s17adc) returns the value of the Bessel function $Y_1(x)$.

2 Specification

```
#include <nag.h>
#include <nags.h>
double nag_bessel_y1 (double x, NagError *fail)
```

3 Description

nag_bessel_y1 (s17adc) evaluates an approximation to the Bessel function of the second kind $Y_1(x)$.

Note: $Y_1(x)$ is undefined for $x \leq 0$ and the function will fail for such arguments.

The function is based on four Chebyshev expansions:

For $0 < x \leq 8$,

$$Y_1(x) = \frac{2}{\pi} \ln x \frac{x}{8} \sum_{r=0} a_r T_r(t) - \frac{2}{\pi x} + \frac{x}{8} \sum_{r=0} b_r T_r(t), \qquad \text{with } t = 2\left(\frac{x}{8}\right)^2 - 1.$$

For $x > 8$,

$$Y_1(x) = \sqrt{\frac{2}{\pi x}} \left\{ P_1(x) \sin\left(x - 3\frac{\pi}{4}\right) + Q_1(x) \cos\left(x - 3\frac{\pi}{4}\right) \right\}$$

where $P_1(x) = \sum_{r=0} c_r T_r(t)$,

and $Q_1(x) = \frac{8}{x} \sum_{r=0} d_r T_r(t)$, with $t = 2\left(\frac{8}{x}\right)^2 - 1$.

For x near zero, $Y_1(x) \simeq -\frac{2}{\pi x}$. This approximation is used when x is sufficiently small for the result to be correct to ***machine precision***. For extremely small x, there is a danger of overflow in calculating $-\frac{2}{\pi x}$ and for such arguments the function will fail.

For very large x, it becomes impossible to provide results with any reasonable accuracy (see Section 7), hence the function fails. Such arguments contain insufficient information to determine the phase of oscillation of $Y_1(x)$; only the amplitude, $\sqrt{\frac{2}{\pi x}}$, can be determined and this is returned on failure. The range for which this occurs is roughly related to ***machine precision***; the function will fail if $x \gtrsim 1/$***machine precision*** (see the Users' Note for your implementation for details).

4 References

Abramowitz M and Stegun I A (1972) *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Clenshaw C W (1962) Chebyshev Series for Mathematical Functions *Mathematical tables* HMSO

5 Arguments

1: **x** – double *Input*

On entry: the argument x of the function.

Constraint: $\mathbf{x} > 0.0$.

2: **fail** – NagError * *Input/Output*

The NAG error argument (see Section 3.6 in the Essential Introduction).

6 Error Indicators and Warnings

NE_INTERNAL_ERROR

An internal error has occurred in this function. Check the function call and any array sizes. If the call is correct then please contact NAG for assistance.

NE_REAL_ARG_GT

On entry, $\mathbf{x} = \langle value\rangle$.
Constraint: $\mathbf{x} \leq \langle value\rangle$.
x is too large, the function returns the amplitude of the Y_1 oscillation, $\sqrt{2/(\pi x)}$.

NE_REAL_ARG_LE

On entry, $\mathbf{x} = \langle value\rangle$.
Constraint: $\mathbf{x} > 0.0$.
Y_1 is undefined, the function returns zero.

NE_REAL_ARG_TOO_SMALL

x is too close to zero and there is danger of overflow, $\mathbf{x} = \langle value\rangle$.
Constraint: $\mathbf{x} > \langle value\rangle$.
The function returns the value of $Y_1(x)$ at the smallest valid argument.

7 Accuracy

Let δ be the relative error in the argument and E be the absolute error in the result. (Since $Y_1(x)$ oscillates about zero, absolute error and not relative error is significant, except for very small x.)

If δ is somewhat larger than the ***machine precision*** (e.g., if δ is due to data errors etc.), then E and δ are approximately related by:

$$E \simeq |xY_0(x) - Y_1(x)|\delta$$

(provided E is also within machine bounds). Figure 1 displays the behaviour of the amplification factor $|xY_0(x) - Y_1(x)|$.

However, if δ is of the same order as ***machine precision***, then rounding errors could make E slightly larger than the above relation predicts.

For very small x, absolute error becomes large, but the relative error in the result is of the same order as δ.

For very large x, the above relation ceases to apply. In this region, $Y_1(x) \simeq \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{3\pi}{4}\right)$. The amplitude $\sqrt{\frac{2}{\pi x}}$ can be calculated with reasonable accuracy for all x, but $\sin\left(x - \frac{3\pi}{4}\right)$ cannot. If $x - \frac{3\pi}{4}$ is written as $2N\pi + \theta$ where N is an integer and $0 \leq \theta < 2\pi$, then $\sin\left(x - \frac{3\pi}{4}\right)$ is determined by θ only. If $x > \delta^{-1}$, θ cannot be determined with any accuracy at all. Thus if x is greater than, or of the order of, the inverse of the ***machine precision***, it is impossible to calculate the phase of $Y_1(x)$ and the function must fail.

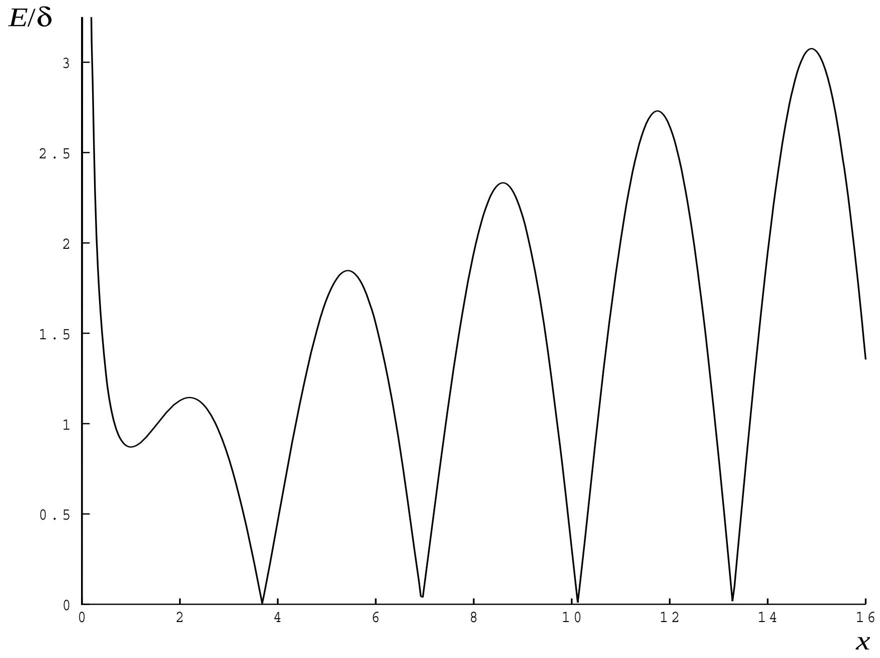


Figure 1

8 Parallelism and Performance

Not applicable.

9 Further Comments

None.

10 Example

This example reads values of the argument x from a file, evaluates the function at each value of x and prints the results.

10.1 Program Text

```
/* nag_bessel_y1 (s17adc) Example Program.
 *
 * Copyright 1990 Numerical Algorithms Group.
 *
 * Mark 2 revised, 1992.
 */

#include <nag.h>
#include <stdio.h>
#include <nag_stdlib.h>
#include <nags.h>

int main(void)
{
  Integer  exit_status = 0;
  double   x, y;
  NagError fail;

  INIT_FAIL(fail);
```

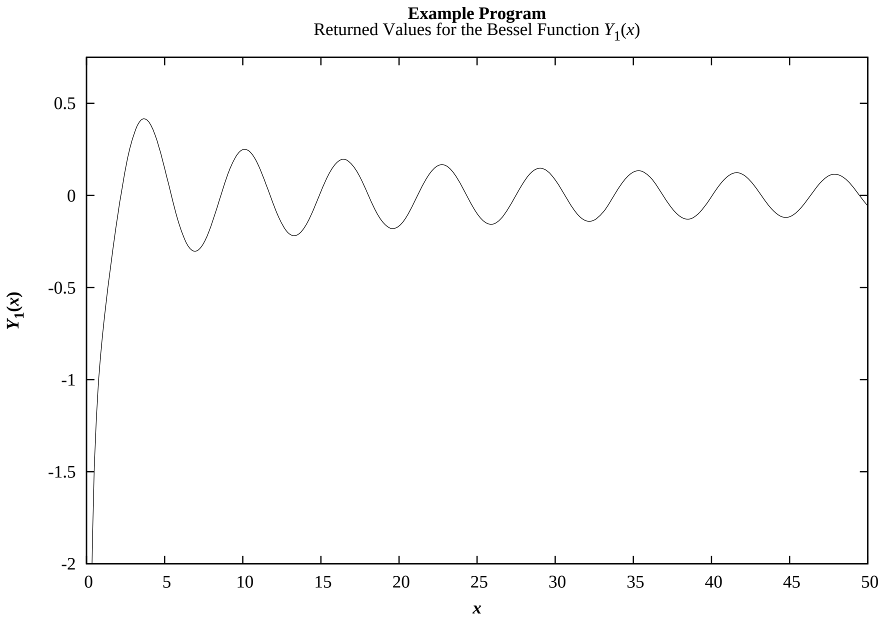
```
  /* Skip heading in data file */
  scanf("%*[^\n]");
  printf("nag_bessel_y1 (s17adc) Example Program Results\n");
  printf("     x           y\n");
  while (scanf("%lf", &x) != EOF)
    {
      /* nag_bessel_y1 (s17adc).
       * Bessel function Y_1(x)
       */
      y = nag_bessel_y1(x, &fail);
      if (fail.code != NE_NOERROR)
        {
          printf("Error from nag_bessel_y1 (s17adc).\n%s\n",
                  fail.message);
          exit_status = 1;
          goto END;
        }
      printf("%12.3e%12.3e\n", x, y);
    }

 END:
  return exit_status;
}
```

10.2 Program Data

```
nag_bessel_y1 (s17adc) Example Program Data
             0.5
             1.0
             3.0
             6.0
             8.0
             10.0
             1000.0
```

10.3 Program Results

```
nag_bessel_y1 (s17adc) Example Program Results
     x           y
   5.000e-01  -1.471e+00
   1.000e+00  -7.812e-01
   3.000e+00   3.247e-01
   6.000e+00  -1.750e-01
   8.000e+00  -1.581e-01
   1.000e+01   2.490e-01
   1.000e+03  -2.478e-02
```


Example Program
Returned Values for the Bessel Function $Y_1(x)$
0.5
0
-0.5
-1
-1.5
-2
$Y_1(x)$
0
5
10
15
20
25
30
35
40
45
50
x